Exercise 47

Show that a solution of $x^6 + 1 = 0$ is $\frac{\sqrt{3}}{2} + \frac{1}{2}i$.

Solution

Let

$$x = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

Then

$$\begin{aligned}
x^6 &= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^6 \\
&= \left[\frac{1}{2}(\sqrt{3} + i)\right]^6 \\
&= \frac{1}{2^6}(\sqrt{3} + i)^6 \\
&= \frac{1}{64}[(\sqrt{3} + i)^3]^2 \\
&= \frac{1}{64}[(\sqrt{3})^3 + 3(\sqrt{3})^2 i + 3(\sqrt{3})i^2 + i^3]^2 \\
&= \frac{1}{64}[\sqrt{27} + 3(3)i + \sqrt{27}(-1) + (-i)]^2 \\
&= \frac{1}{64}(\sqrt{27} + 9i - \sqrt{27} - i)^2 \\
&= \frac{1}{64}(8i)^2 \\
&= \frac{1}{64}(64i^2) \\
&= i^2 \\
&= -1
\end{aligned}$$

Therefore, adding 1 to both sides,

$$x^6 + 1 = 0.$$